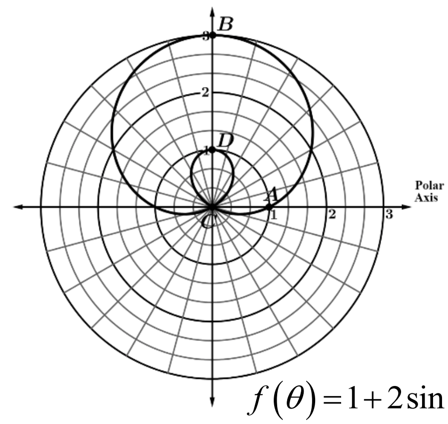
In this section, we will learn ways that we can describe characteristics of the graph of a polar function.

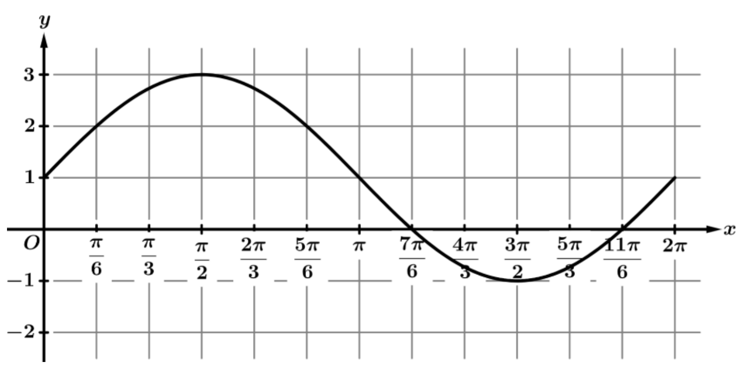
Given a polar function $r = f(\theta)$, we know that $r$ represents the "signed radius" of the function. We use the phrase "signed radius" because $r$ can be a positive or negative value.

As we trace the graph of a polar function, we are interested in whether the graph of $r = f(\theta)$ is getting closer to the origin or further from the origin over a given interval.

<table>
<tr><th colspan="3">Changes in the Distance from $r = f(\theta)$ to the Origin</th></tr>
<tr><td>$r = f(\theta)$ is <u>positive</u> and <u>increasing</u></td><td rowspan="2">The distance between $r = f(\theta)$ and the origin is <u>increasing</u></td><td rowspan="4">

$f(\theta) = 1 + 2\sin\theta$</td></tr>
<tr><td>$r = f(\theta)$ is <u>negative</u> and <u>decreasing</u></td></tr>
<tr><td>$r = f(\theta)$ is <u>positive</u> and <u>decreasing</u></td><td rowspan="2">The distance between $r = f(\theta)$ and the origin is <u>decreasing</u></td></tr>
<tr><td>$r = f(\theta)$ is <u>negative</u> and <u>increasing</u></td></tr>
</table>

**Example 1:** The graph of $f(x) = 1 + 2\sin x$ is shown below for $0 \le x \le 2\pi$. Use the graph below to complete the given table with the appropriate intervals.




| Description of $f(x)$ | Interval(s) |
|---|---|
| $f$ is **positive** and **increasing** | |
| $f$ is **positive** and **decreasing** | |
| $f$ is **negative** and **increasing** | |
| $f$ is **negative** and **decreasing** | |

In the polar coordinate system, the graph of $f(x) = 1 + 2\sin x$ above becomes $f(\theta) = 1 + 2\sin\theta$, as shown in the table above. The labeled points A, B, C, and D correspond to the intervals found in **Example 1**.

| Points on $f(\theta)$ | From A to B | From B to C | From C to D | From D to C | From C to A |
|---|---|---|---|---|---|
| Interval | $0 < \theta < \frac{\pi}{2}$ | $\frac{\pi}{2} < \theta < \frac{7\pi}{6}$ | $\frac{7\pi}{6} < \theta < \frac{3\pi}{2}$ | $\frac{3\pi}{2} < \theta < \frac{11\pi}{6}$ | $\frac{11\pi}{6} < \theta < 2\pi$ |
| $r = f(\theta)$ is | **positive** and **increasing** | **positive** and **decreasing** | **negative** and **decreasing** | **negative** and **increasing** | **positive** and **increasing** |
| Distance between $f(\theta)$ and the origin is | increasing | decreasing | increasing | decreasing | increasing |

**AP Exam Tip:** It is often helpful to sketch the graph of a given function in rectangular coordinates when attempting to describe the behavior of a polar function.

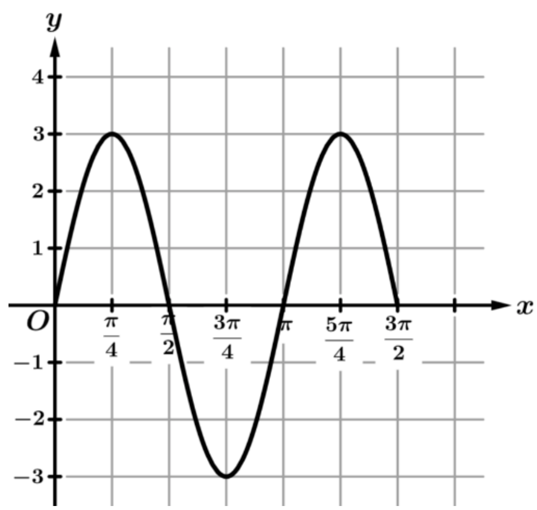
**Example 2:** Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 2 - 4\cos\theta$, in the polar coordinate system for $0 \le \theta \le 2\pi$. Which of the following statements is true about the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin?

(A) The distance is increasing for $\pi < \theta < \frac{5\pi}{3}$, because $f(\theta)$ is positive and increasing on the interval.

(B) The distance is increasing for $\frac{5\pi}{3} < \theta < 2\pi$, because $f(\theta)$ is negative and increasing on the interval.

(C) The distance is decreasing for $\pi < \theta < \frac{5\pi}{3}$, because $f(\theta)$ is positive and decreasing on the interval.

(D) The distance is decreasing for $\frac{5\pi}{3} < \theta < 2\pi$, because $f(\theta)$ is negative and decreasing on the interval.




Graph of $f(x)$

**Example 3:** The graph of $f(x) = 3\sin(2x)$, where $0 \le x \le \frac{3\pi}{2}$ is shown above in the rectangular coordinate system.

If the polar function $r = f(\theta)$, where $f(\theta) = 3\sin(2\theta)$, is graphed in the polar coordinate system for $0 \le \theta \le \frac{3\pi}{2}$, on which of the following intervals is the distance between the point with polar coordinates $(f(\theta), \theta)$ and the origin decreasing?

(A) $0 < \theta < \frac{\pi}{4}$ (B) $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$ (C) $\frac{3\pi}{4} < \theta < \pi$ (D) $\pi < \theta < \frac{5\pi}{4}$

## Relative Extrema and Polar Functions

Another characteristic that arises when we study polar functions are relative extrema (minima and maxima). For polar functions, if $r = f(\theta)$ changes from increasing to decreasing (or from decreasing to increasing), then the function has a relative extremum on the interval corresponding to a point relatively closest to or farthest from the origin.

**Example 4:** Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 1 - 2\sin(2\theta)$, in the polar coordinate system for $0 \le \theta \le \pi$. Which of the following statements is true about the graph of $r = f(\theta)$?

(A) The graph of $r = f(\theta)$ has a relative minimum on the interval $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, because $r = f(\theta)$ changes from negative to positive.

(B) The graph of $r = f(\theta)$ has a relative minimum on the interval $\frac{2\pi}{3} < \theta < \pi$, because $r = f(\theta)$ changes from decreasing to increasing.

(C) The graph of $r = f(\theta)$ has a relative maximum on the interval $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, because $r = f(\theta)$ changes from positive to negative.

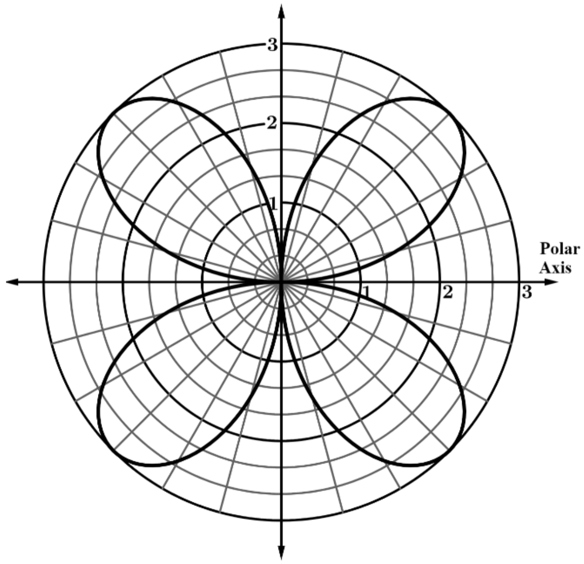
(D) The graph of $r = f(\theta)$ has a relative maximum on the interval $\frac{2\pi}{3} < \theta < \pi$, because $r = f(\theta)$ changes from increasing to decreasing.

## Average Rate of Change

In previous units, we learned about the average rate of change of a function in rectangular coordinates. In the polar coordinate system, we will find the average rate of change of $r$ with respect to $\theta$ over a given interval of $\theta$.

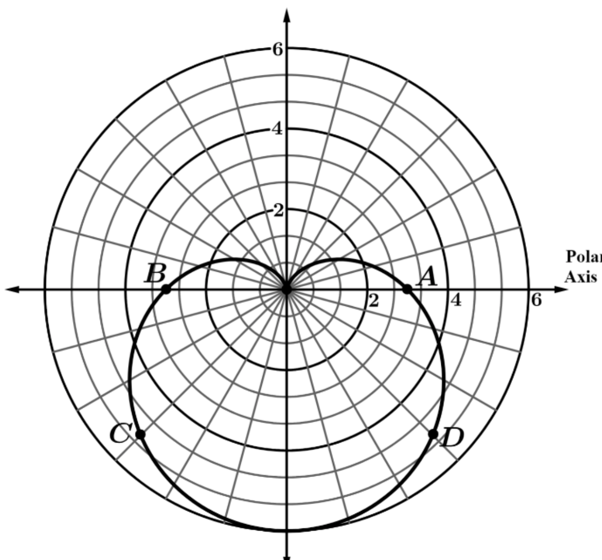
| Average Rate of Change of a Polar Function | |
|---|---|
| For the polar function $r = f(\theta)$, the average rate of change of $r = f(\theta)$ over the interval $a \le \theta \le b$ is given by the expression $\frac{f(b) - f(a)}{b - a}$. | Geometrically, the average rate of change indicates the rate at which the radius is changing per radian. |

**Example 5:** Consider the graph of the polar function $r = f(\theta)$, where $f(\theta) = 3 - 3\cos\theta$, in the polar coordinate system. What is the average rate of change of $r = f(\theta)$ over the interval $\frac{\pi}{2} \le \theta \le \pi$?




**Example 6:** The figure shows the graph of the polar function $r = f(\theta)$, where $f(\theta) = 3\sin(2\theta)$ for $0 \le \theta \le 2\pi$, in the polar coordinate system. On which of the following intervals is the average rate of change of $f(\theta)$ equal to zero?

(A) $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$ (B) $\frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}$ (C) $\frac{3\pi}{4} \le \theta \le \frac{7\pi}{4}$ (D) $\frac{\pi}{4} \le \theta \le \frac{7\pi}{4}$




**Example 7:** In the polar coordinate system, the graph of a polar function $r = f(\theta)$ is shown with a domain of all real values of $\theta$ for $0 \le \theta \le 2\pi$. On this interval of $\theta$, the graph has no holes, passes through each point exactly one time, and as $\theta$ increases, the graph passes through the labeled points A, B, C, and D, in that order. On which of the following intervals is the average rate of change of $r$ with respect to $\theta$ least?

(A) From A to B

(B) From B to C

(C) From C to D

(D) From D to A

**Estimating Values of $r = f(\theta)$ Using the Average Rate of Change**

For a given interval, we can use the average rate of change of $r = f(\theta)$ over the interval to estimate other values of $r = f(\theta)$ inside the given interval.

**Recall**: The point-slope form of a line is given by $y - y_1 = m(x - x_1)$, where $(x_1, y_1)$ is a known point on the line and $m$ is the slope of the line.

We can utilize this concept and create a linear function that will help us approximate a given polar function. We will use the average rate of change of $r$ with respect to $\theta$ as our slope, and we can use a point at either end of the given interval as our given point. This leads us to the following

$$f(\theta) \approx f(\theta_1) + \frac{f(b) - f(a)}{b - a}(\theta - \theta_1),$$

where $(f(\theta_1), \theta_1)$ is a known point on the graph of $r = f(\theta)$, and the average rate of change of $r = f(\theta)$ with respect to $\theta$ over the interval $a \le \theta \le b$ is given by $\dfrac{f(b) - f(a)}{b - a}$.

| $\theta$ | $\frac{\pi}{6}$ | $\frac{7\pi}{6}$ |
|---|---|---|
| $f(\theta)$ | 2 | $-1$ |

**Example 8:** The table above gives values of the polar function $r = f(\theta)$ at selected values of $\theta$. Use the average rate of change of $r = f(\theta)$ over the interval $\frac{\pi}{6} \le \theta \le \frac{7\pi}{6}$ to approximate $f\left(\frac{5\pi}{6}\right)$.